On the quasi-steady aerodynamics of normal hovering flight part I: the induced power factor

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An analytical treatment to quantify the losses captured in the induced power factor, $k$, is provided for flapping wings in normal hover, including the effects of non-uniform downwash, tip losses and finite flapping amplitude. The method is based on a novel combination of actuator disc and lifting line blade theories that also takes into account the effect of advance ratio. The model has been evaluated against experimental results from the literature and qualitative agreement obtained for the effect of advance ratio on the lift coefficient of revolving wings. Comparison with quantitative experimental data for the circulation as a function of span for a fruitfly wing shows that the model is able to correctly predict the circulation shape of variation, including both the magnitude of the peak circulation and the rate of decay in circulation towards zero. An evaluation of the contributions to induced power factor in normal hover for eight insects is provided. It is also shown how Reynolds number can be accounted for in the induced power factor, and good agreement is obtained between predicted span efficiency as a function of Reynolds number and numerical results from the literature. Lastly, it is shown that for a flapping wing in hover $k$ owing to the non-uniform downwash effect can be reduced to 1.02 using an arcsinh chord distribution. For morphologically realistic wing shapes based on beta distributions, it is shown that a value of 1.07 can be achieved for a radius of first moment of wing area at 40% of wing length.

1. Introduction

The ability for an animal to achieve sustained flight allows much greater exploitation of their environment compared with equivalent animals that cannot fly. For smaller animals, particularly insects, the ability to hover may be a critical part of food gathering. It is well known that hovering is an energetically very expensive form of locomotion, and hence understanding of the aerodynamics of flapping flight in hover, and the power cost associated with weight support, is an important area of study.

The actuator disc theory provides a simple momentum-based model for the interaction of a revolving (rotary) or flapping wing with a surrounding fluid based on the assumption that the wing induces a uniform downwash velocity over the area swept by the wing. This assumption is consistent with minimization of the induced power required for flight; however, in reality, the downwash is not uniform, and the induced power required is increased [1]. The ratio of actual induced power to minimum ideal induced power for a given thrust is known as the induced power factor, $k$. This paper provides a first principle approach to determine $k$ in order to provide improved understanding of the aerodynamic performance of a range of different insects in hover, and to provide support for the development of engineering tools for optimization of the wing planforms and kinematics of insect scale flapping wing vehicles. While the study is focused on a correction to the ideal induced power, this correction may be as high as a factor of two, and hence forms a fundamental part of the aerodynamic treatment.

A significant amount of work on flapping animals has been undertaken with the aim of identifying wing inviscid span efficiency (inverse of induced
power factor \([2,3]\) in forward flight through experiments \([1,4–8]\), with the downwash velocity distribution measured using digital particle image velocimetry (DPIV) techniques. These data are then used within the actuator disc theory framework to define the real lift and induced power values. Comparison with ideal conditions then allows the evaluation of the induced power factor and hence flapping wing inviscid span efficiency.Henningsson & Bomphrey \([4]\) obtained a maximum span efficiency within the flapping cycle of forward flying locusts of 0.79 and an average span efficiency value of 0.53, implying \(k\)-values of 1.27 and 1.89, respectively. Recently, Henningsson & Bomphrey \([5]\) assessed the span efficiency of six hawkmoth species flying tethered in a wind tunnel. The obtained average span efficiencies for the moths ranged from 0.31 to 0.6; equivalent to \(k\)-values ranging from 1.67 to 3.23. Muijres \textit{et al}. \([6,7]\) performed similar measurements on bats and pied flycatchers; however, they used a hovering-induced power factor expression to evaluate the \(k\)-values at different speeds. For bats, they obtained an average \(k\)-value throughout the measured speed range of 1.25. Average \(k\)-values between 1.1 and 1.25 were obtained for pied flycatchers. In addition, Johansson \textit{et al}. \([8]\) followed the same methodology in \([6,7]\) and obtained an average \(k\)-value of 1.61 for flying beetles.

An analytical approach to determine the induced power factor for hovering flight is provided by Ellington \([9]\), who used a combined modified actuator disc and vortex theory to give so-called temporal and spatial corrections for the induced power. He showed that each correction is around 10\%, giving an overall \(k\)-value around 1.2. The temporal correction is related to tip losses, whereas the spatial correction accounts for non-uniformities over the ideal actuator disc. Later, Sunada & Ellington \([10]\) proposed a more sophisticated method for the evaluation of \(k\) in which they modelled flapping forces with the added masses of vortex wake sheets. This analysis gave normal hovering \(k\)-values between 1.2 and 1.4 for the different species investigated.

The aim of this paper is to provide a transparent analytical treatment to capture the different aerodynamic effects influencing a real flapping wing in normal hovering flight using the single parameter, \(k\). It builds upon previous analytical treatments through accounting for contributions to \(k\) owing to both wing shape and flapping kinematics. Section 2 develops a model for the induced power losses in normal hovering owing to non-uniform downwash velocity distribution, tip losses and finite flapping amplitude. Section 3 demonstrates the validity of the approach by providing a comparison of model results with experiments in the literature; an evaluation of the induced power factor in normal hover for eight insects is then provided. In §4, the developed model is used to understand the effect of Reynolds number on induced power factor, and then used to identify optimum wing geometry and kinematics for minimizing induced power.

2. Contributions to the induced power factor

2.1. Analytical model for induced power losses

The method for modelling normal hovering flapping flight proposed here is based on the method of Stepniewski & Keys \([11]\) for evaluation of the induced power factor of hovering rotors, with appropriate modifications applied to represent flapping flight. The flapping wing system is approximated by an actuator disc of area \(S_d\). The mass flow rate, \(m\), of air through the disc is

\[
\dot{m} = \int_{S_d} \rho \omega dS_d, \tag{2.1}
\]

where \(\rho\) is the air density and \(\omega\) is the downwash velocity. For normal hovering flapping flight, the effective disc area is the area of sectors swept out by both flapping wings and is given by \([9,12]\)

\[
S_d = \Phi R^2, \tag{2.2}
\]

where \(\Phi\) is the flapping stroke angle and \(R\) is the wing length. It should be noted that equation (2.2) may appear inconsistent with the use of a circular actuator area for non-flapping wings. However, it can be argued that the flapping case is sufficiently different to other modes of flight that a different definition of the actuator disc area must be used. In particular, the wake dimensions for flapping flight must be influenced by the stroke angle, \(\Phi\). Thus, it is more appropriate to define \(S_d\) as the area over which the wings actually impart downward momentum to the air \([9]\). Following equation (2.2), an elementary disc area is

\[
dS_d = 2\Phi R dr, \tag{2.3}
\]

and substitution into (2.1) gives

\[
\dot{m} = \int_{0}^{R} \rho \omega(r) 2\Phi R dr. \tag{2.4}
\]

Note that the upper bound of the above integration is changed to the effective radius, \(R_o\), to account for the aerodynamic phenomena occurring at the outer rim of the disc reducing its lift generating effectiveness in that region generating so-called tip losses. The generated lift force, \(L\) (usually called thrust for rotors) from the actuator disc is equal to the rate of change of downward momentum, which is obtained by multiplying the mass flow rate by the eventual downwards velocity, which is equal to twice the induced velocity at the disc

\[
L = 4\Phi \rho \int_{0}^{r} (\omega(r))^2 r dr. \tag{2.5}
\]

Expression (2.5) can be written as

\[
L = 4\Phi R^2 \rho \int_{0}^{\hat{r}} (\omega(\hat{r}))^2 \hat{r} d\hat{r}, \tag{2.6}
\]

where \(\hat{r} = r/R\) and \(B\) is the non-dimensional effective radius (\(R_o/R\)). The corresponding induced power is

\[
P_{ind} = 4\Phi R^2 \rho \int_{0}^{\hat{r}} (\omega(\hat{r}))^2 \hat{r} d\hat{r}. \tag{2.7}
\]

On the other hand, if a constant downwash velocity distribution is achieved, there are no tip losses and the wings sweep the maximum possible disc area (i.e. \(S_d = \pi R^2\)), the ideal lift produced is

\[
L_{ideal} = 2\pi R^2 \rho \omega^2, \tag{2.8}
\]

and

\[
P_{ideal} = 2\pi R^2 \rho \omega^3. \tag{2.9}
\]

The induced power factor, \(k\), is obtained as the quotient \(P_{ind}/P_{ideal}\), where the uniform ideal downwash velocity required within the ideal power expression (equation (2.9)) is obtained by equating equations (2.6) and (2.8). This leads...
The first contributor to the overall induced power factor expression, \( k_{\text{ind}} \), considers the effect of non-uniform downwash distribution, and a substantial discussion is provided on the effect of chord distribution, advance ratio and root offset on this term in §2.2. The second contributor to the overall \( k \) considers tip losses and is referred to as \( k_{\text{tip}} \). It will be discussed in the context of its derivation from the ‘finite number of blades’ concept from rotary wing aerodynamics as well as its derivation from Ellington’s temporal correction. Finally, the term \( k_{\text{flap}} \) is based on simple geometrical considerations and will be discussed briefly at the end. The three sources of inefficiency are schematically represented in figure 1.

2.2. Non-uniform downwash velocity effect

Here, the effect of non-uniform downwash velocity is discussed. Equation (2.11) can be used to evaluate \( k_{\text{ind}} \) if the induced downwash velocity distribution over the wing is known. We propose here an analytical method based on lifting line blade models. Sane [13] presented a lifting line blade model for hovering flapping wings; however, the model relied on empirical experimental data, and hence measurements are still required for the calculation. Leishman [14, ch. 14] provided a generic formulation of the lifting line problem. In addition, Ansari et al. [15] discussed such models in the context of insect-like flapping wings.

### Figure 1

A schematic of the three sources of inefficiency within flapping normal wash distribution is discussed in §2.2. (\( \text{Online version in colour.} \))

### Figure 2

Sectional flow velocities of a flapping wing with an additional translational constant velocity component.

\[
V(r) = \phi r + V_t \cos \phi, \quad (2.14)
\]

where \( \phi \) is flapping angle amplitude. The wing tip angular flapping velocity is given the name \( V_{\text{tip}} \).

\[
V_{\text{tip}} = \phi R. \quad (2.15)
\]

\( J_1 \) is defined as the ratio of the chordwise components of the wing tip flow velocity owing to translation and revolution

\[
J_1 = \frac{V_t \cos \phi}{V_{\text{tip}}}. \quad (2.16)
\]

Therefore, over one flapping cycle, \( J_1 \) will vary between \(-J\) and \( J \) [16], where \( J \) is the advance ratio given by

\[
J = \frac{V_t}{|V_{\text{tip}}|}. \quad (2.17)
\]

## Figure 3

A schematic of a flapping wing offset.

However, in the last two references, generic formulae were only provided without complete details of the method. Here, a more general formulation of the lifting line problem is presented.

As a starting point, the velocity distribution, \( V(r) \), on the wing must be defined. The more general case of a wing moving with an angular velocity, \( \omega \), and additionally experiencing a constant free stream velocity component, \( V_f \), parallel to the flapping plane is shown in figure 2. For this case, the sectional flow velocity at a station \( r \) from the centre of rotation is given by [16]

\[
V(r) = \phi r + V_t \cos \phi, \quad (2.14)
\]
linear varying velocity; hence, in effect, it can be treated in the same manner as the advance ratio, where $J_1$ is defined here as

$$J_1 = \frac{\partial R_i}{\partial R} = \frac{R_i}{R} - \frac{R_i}{R_0} - \frac{R_i}{R_0}.$$  \hfill (2.19)

The wing offset will have an additional effect on the induced power factor as a root cut out that reduces the effective disc area; however, to maintain simplicity, this effect is not considered here.

Now, the wing can be modelled as a vortex of strength $\Gamma(r)$ bound to the aerodynamic centre and the lift per unit span can be obtained using the Kutta–Joukowski theorem as [14]

$$dL = \rho V(r)\Gamma(r)\,dr = \frac{1}{2} \rho V(r)^2 c(r)\,dr C_{L,2D}(\alpha - \alpha_i).$$  \hfill (2.20)

where $c$ is the chord, $C_{L,2D}$ is the two-dimensional-aerofoil lift–curve slope, $\alpha$ is the angle of attack and $\alpha_i$ is the induced angle of attack. Hence, $\Gamma(r)$ can be obtained as

$$\Gamma(r) = \frac{L}{\rho V(r)^2 c(r)} C_{L,2D}(\alpha - \alpha_i).$$  \hfill (2.21)

The distribution of the induced downwash velocity along the wing length, $w(r)$, can be obtained by applying the Biot–Savart law to the vortex wake produced by the wing [17,18]

$$w(r) = \frac{1}{4 \pi} \int_{-R}^{r} \frac{\Gamma(r')\,dr'}{r'^2 - r^2},$$  \hfill (2.22)

where $r$ is the selected wing location at which the downwash velocity is required, and $r$ is the location of vortices causing the downwash [17]. In the above relations, a wing location can be replaced with [17]

$$r = -R \cos \theta,$$  \hfill (2.23)

where $\theta$ is now used to define position along the wing. In addition, the vortex strength, $\Gamma(r)$, is written in a non-dimensional form as [17,18]

$$\gamma = \frac{\Gamma(r)}{2\pi V(r)} = \sum_{m=1}^{\infty} a_m \sin m \theta;$$  \hfill (2.24)

hence

$$\Gamma(r) = 4\pi V(r) \sum_{m=1}^{\infty} a_m \sin m \theta.$$  \hfill (2.25)

Substituting the velocity distribution of equation (2.14) (using equations (2.16) and (2.23)) into the equation for circulation (2.25) leads to

$$\Gamma(r) = 4V_{tip}R \left( J_1 \left( \sum_{m=1}^{\infty} a_m \sin m \theta \right) - \left( \cos \theta \sum_{m=1}^{\infty} a_m \sin m \theta \right) \right).$$  \hfill (2.26)

Substituting equation (2.26) into equation (2.22) and performing integration using the Glaucut integrals [19] leads to the following expression for the downwash:

$$w(r) = V_{tip} \left( J_1 \sum_{m=1}^{\infty} m a_m \frac{\sin m \theta}{\sin \theta} \right) - \left( \sum_{m=1}^{\infty} m a_m \frac{\cos \theta \sin m \theta}{\sin \theta} + \sum_{m=1}^{\infty} a_m \cos m \theta \right).$$  \hfill (2.27)

The $a_m$ coefficients in equation (2.27) can be obtained by equating equations (2.21) and (2.26) using equations (2.14) and (2.27), leading to

$$\mu \alpha (J_1 \sin \theta - \sin \theta \cos \theta) = J_1 \sum_{m=1}^{\infty} a_m \sin \theta \sin m \theta$$

$$- \sum_{m=1}^{\infty} a_m \sin \theta \cos \theta \sin m \theta + \mu \left( J_1 \sum_{m=1}^{\infty} m a_m \sin m \theta \right)$$

$$- \sum_{m=1}^{\infty} a_m \sin \theta \cos \theta \sin m \theta - \sum_{m=1}^{\infty} m a_m \cos \theta \sin m \theta).$$  \hfill (2.28)

where $\mu = c(\alpha)C_{L,2D}/8R$. For the two-dimensional lift–curve slope, $C_{L,2D}$, either the experimental (0.09 deg)$^{-1}$ or the theoretical value (2$\pi$ rad$^{-1}$ = 0.11 deg$^{-1}$) for a flat plate can be used. Calculated values of $k_{rad}$ are relatively insensitive to lift–curve slope value, and it is usual practice to use the experimental value. In this study, a value of 0.09 deg$^{-1}$ is used which is based on the experimental work of Okamoto et al. [20], who obtained this value for a flat plate at typical insects Reynolds numbers. Owing to the symmetry of load distribution, only the odd terms of $m$ are considered. Following the conventional lifting line solution procedure, the series is truncated at a convenient number of terms, and the equation (2.28) is satisfied at a number of wing stations resulting in a set of simultaneous linear equations. This set is solved for the $a_m$ coefficients and hence the downwash distribution is obtained. In the limit when $J_1$ approaches infinity, the well-known monoplane fixed wing equations are obtained, whereas if $J_1$ is zero, the typical normal hovering case is simulated. Therefore, equations (2.26), (2.27) and (2.28) represent a more general formulation of the lifting line problem. It should be noted that the $k_{rad}$ value is sensitive to the chord distribution and advance ratio. Therefore, $k_{rad}$ depends on the wing morphology (through chord distribution) and kinematics (through advance ratio). Further discussion of these effects is provided in the following sections of this work.

### 2.3. Tip loss effect

It is well known that lifting line blade models are unable to fully capture the flow structure at the blade/wing tip (see discussion in reference [21, ch. 10]). There are a number of options for correction for corrected tip effects ranging from simple tip loss factors to a complete lifting surface theory analysis of the aerodynamics. For this work, we adopt a tip loss factor approach that models the loss as an effective reduction in wing span.

Prandtl provided a solution for a tip loss correction for a rotor with finite number of blades. He showed that when accounting for tip loss, the effective blade radius, $R_{e}$, is given by [14,21]

$$B = \frac{R_e}{R} = 1 - \left( \frac{2\ln^2 N_b}{N_b} \right) \frac{\lambda}{\sqrt{1 + \lambda^2}}.$$  \hfill (2.29)

where $N_b$ is the number of blades and $\lambda$ is the inflow ratio. Although this formula was originally developed for rotors with finite number of blades, Sane [13] showed that it can be used within the context of flapping flight, suggesting a value of 2 for $N_b$ to simulate a complete wing cycle and a value of 1 for a single up- or downstroke. For a hovering case, $\lambda$ is the ratio of the induced downwash velocity to the wing tip velocity.
and can be evaluated using the simple model \[11, 14\]

\[
\lambda = \frac{1}{V_{\text{tip}}} \sqrt{\frac{\text{DL}}{2\rho}},
\]

(2.30)

where DL is disc loading obtained as the quotient of the thrust to the effective disc area.

A different tip loss model is provided by Ellington \[9\], where a temporal correction is applied to the Rankine–Froude theory to account for wake periodicity. Ellington stated that his model compares well with hovering helicopters. It will be shown in §3 that the two methods are in very good numerical agreement. Nevertheless, he also added that his method provides a more satisfying physical and conceptual description of the flapping problem; hence, it will be presented here in some detail. The expression for the temporal correction is given as \[9\]

\[
k_{\text{tip}, \text{Ellington}} = 1 + 0.079s^2,
\]

(2.31)

where \(s\) is the so-called spacing parameter, which for a normal hovering case with a horizontal stroke plane is given by \[9\]

\[
s = \frac{2\pi W}{\mu v^2 (\Phi R^2)^2},
\]

(2.32)

where \(W\) is the insect weight, \(f\) is the wing beat frequency and \(n\) is the frequency of lift impulses per wing beat frequency. It takes a value of 2 in normal hovering as the two wing strokes provide weight support and takes a value of 1 if only one wing stroke provides weight support. Therefore, the number of lift impulses per wing beat frequency of Ellington’s model is analogous to the number of blades of the rotor model.

2.4. Finite flapping amplitude effect

The last contributor to the overall induced power factor expression, \(k_{\text{flap}}\), considers losses associated with the reduction of effective disc area for flapping stroke angles less than 180°. Hence, this reduction in the effective disc area will cause further increase in the disc loading and higher induced velocity compared with that given by the simple momentum theory, leading to a third contributor to the overall induced power factor. We name it \(k_{\text{flap}}\) as it is an induced power loss specific to flapping wings only.

3. Results

3.1. Comparison with experiments

Here, the model for induced power factor proposed above will be tested against available experimental measurements from the literature. First, the model is compared with the experimental study of Dickson & Dickinson \[16\], in which they have investigated the effect of advance ratio on the aerodynamics of revolving wings. A wing with \(R = 0.25\) m, \(AR = 4.2\) and a non-dimensional radius of the first moment of wing area of 0.59 was used. Experimental measurements of lift and drag coefficients were provided for a range of kinematics corresponding to \(J_1\) ranging from –0.5 to 0.5. One of the main conclusions was that there is a negative correlation between the lift coefficient and \(J_1\). Here, we used their same experimental conditions within the proposed model for calculating the downwash velocity distribution, which was then used within equation (2.11) to evaluate \(k_{\text{flap}}\) and this was repeated for the different \(J_1\) values of the experiment. The wing chord distribution was defined based on the beta function proposed by Ellington \[22\] (this function is presented in §3.2).

Results for the variation of \(k_{\text{ind}}\) over the \(J_1\) range of the experiment (–0.5 < \(J_1\) < 0.5) are shown in figure 4. On the basis that by definition the lift coefficient is inversely proportional to the k-factor, the obtained results provide qualitative agreement with the experimental observations of Dickson and Dickinson.

Next, we compare our model with results from Birch et al. \[23\], who used DPIV to measure the circulation around a revolving model fruitfly wing with a mean chord of 7 cm \[13\] in a typical hovering condition (\(J_1 = 0\)). Because the wing has a planform shape of a fruitfly wing, we used the real fruitfly AR of 3.015 and non-dimensional radius of the first moment of wing area of 0.55 (table 1) to define the chord distribution based on the beta function. The measurements were performed with the wing set at a 45° geometrical angle of attack and revolved with a wing tip velocity of 0.26 m s\(^{-1}\). The circulation distribution at the same experimental conditions was calculated using the lifting line model, and the result was compared with the experiment (figure 5). In figure 5, the semi-empirical treatment of Sane \[13\] for the same experiment is included for comparison. Excellent agreement between analytical model and experiment for the shape of variation, including both the peak value of circulation and the rate of decay of circulation towards zero is found. Note that the lifting line model assumes a totally uniform system and does not include a tip loss effect as a result of wake periodicity; hence, prediction of the effective wing tip location is not an explicit output of the model. As will be shown in the next section, for a revolving fruitfly wing, we get a value of \(B\) (inverse of \(k_{\text{flap}}\)) of 0.87 which is in a very good agreement with the experimental results.

3.2. Induced power factor values

Here, the value of the different contributors to the induced power factor for eight insects in normal hovering with \(J_1 = 0\) is presented. Table 1 shows weight, morphological and kinematic data of the eight insects taken from Sun & Du \[24\]. Note that these data were collected by Sun and Du from the most relevant study for each insect.

In this evaluation, the chord distribution for the different insects was defined based on the method proposed by Ellington \[22\]. For many insect wings, Ellington found that the chord distribution is accurately described to within 5% of the measured values using a beta function as \[22\]

\[
c(r) = c \left(\frac{r^{\alpha-1}(1 - r)^{\beta-1}}{\beta!} \right),
\]

(3.1)
Table 1. Weight, morphological and kinematic parameters of hovering insects [24].

<table>
<thead>
<tr>
<th>insect</th>
<th>mass (mg)</th>
<th>$R$ (mm)</th>
<th>$c$ (mm)</th>
<th>$\bar{r}_1$</th>
<th>$f$ (Hz)</th>
<th>$\Phi$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.02</td>
<td>0.67</td>
<td>0.55</td>
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<td>150</td>
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<tr>
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<td>4.02</td>
<td>0.49</td>
<td>155</td>
<td>116</td>
</tr>
<tr>
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<td>51.9</td>
<td>18.26</td>
<td>0.46</td>
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<tr>
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<td>9.8</td>
<td>3.08</td>
<td>0.5</td>
<td>197</td>
<td>131</td>
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<tr>
<td>cranfly</td>
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<td>12.7</td>
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<td>0.56</td>
<td>45.5</td>
<td>123</td>
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<td>3.23</td>
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4. Applications and discussion

4.1. Reynolds number effect on the induced power factor

There is currently some evidence that suggests that hovering flapping wing flows at insect scale are largely independent of Reynolds number. This is consistent with the experimental observations that shows that the net force vector at high angles of attack is normal to the wing surface indicating dominance of pressure forces at these low Reynolds numbers and the relative lack of influence of viscous forces [23, 27, 28]. In an experimental study, Lentink & Dickinson [29] tested the aerodynamics of a fruitfly wing model at three Reynolds numbers using the same kinematics, but using fluids of different viscosities. They found that the lift–drag coefficient polars did not change much and almost no dependence on Reynolds number over the range they measured (100 < Re < 14 000) was demonstrated. Similarly, Sane & Dickinson [30] reported that their measured forces might not be crucially dependent on viscosity, stating that ‘Both viscid and inviscid models show reasonable agreement with forces measured on our apparatus using identical kinematics’ [30, p. 1094].

Building on the previous, a proposed analytical manipulation that would explain the variation of the induced power factor with Reynolds number is presented. This will be achieved using the analytical expression of the induced power factor owing to tip loss provided by Ellington (equation (2.31)).
characteristics of flapping wings discussed previously. However, this can be accepted at typical insects Reynolds number based on the wing tip speed and chord, whereas the viscosity has no effect. The Reynolds number adopted from [32] is included for comparison. (Online version in colour.)

Table 2. Calculated contributions to the induced power factor, $k$.

<table>
<thead>
<tr>
<th>insect</th>
<th>$k_{\text{ind}}$</th>
<th>$k_{\text{tip}}$ (equation (2.29) with $S_d = \pi R^2$)</th>
<th>$k_{\text{flap}}$</th>
<th>$k_{\text{ind}}$</th>
<th>$k_{\text{tip}}$</th>
<th>$k_{\text{flap}}$ (equation (2.31))</th>
</tr>
</thead>
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<td>fruitfly</td>
<td>1.21</td>
<td>1.14</td>
<td>1.10</td>
<td>1.51</td>
<td>1.11</td>
<td></td>
</tr>
<tr>
<td>bumble-bee</td>
<td>1.17</td>
<td>1.10</td>
<td>1.25</td>
<td>1.61</td>
<td>1.07</td>
<td></td>
</tr>
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<td>hawkmoth</td>
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<td>1.12</td>
<td>1.22</td>
<td>1.55</td>
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</tr>
<tr>
<td>honeybee</td>
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<td>1.10</td>
<td>1.17</td>
<td>1.52</td>
<td>1.07</td>
<td></td>
</tr>
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<td>1.88</td>
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<tr>
<td>dronefly</td>
<td>1.16</td>
<td>1.06</td>
<td>1.29</td>
<td>1.57</td>
<td>1.03</td>
<td></td>
</tr>
<tr>
<td>ladybird</td>
<td>1.15</td>
<td>1.12</td>
<td>1.01</td>
<td>1.30</td>
<td>1.09</td>
<td></td>
</tr>
<tr>
<td>mean ± s.d.</td>
<td>1.18 ± 0.034</td>
<td>1.1 ± 0.024</td>
<td>1.21 ± 0.12</td>
<td>1.57 ± 0.16</td>
<td>1.074 ± 0.024</td>
<td></td>
</tr>
</tbody>
</table>

Here, we propose rearranging equation (2.32) for the spacing parameter as follows

$$s^2 = \frac{2\pi W}{\rho} \left( \frac{1}{\AR^2} \right) \left( \frac{1}{2\Phi R c} \right)^2$$

where $\AR$ is a single wing aspect ratio, $2\Phi R c$ is the mean wing tip speed, $c$ is the mean chord, $\nu$ is the air kinematic viscosity and $Re$ is the Reynolds number based on the wing tip speed [31]. In equation (4.1), the kinematic viscosity was enforced into the expression to obtain the Reynolds number. This means that the main parameter controlling the Reynolds number is the wing speed and chord, whereas the viscosity has no effect. However, this can be accepted at typical insects Reynolds numbers owing to the minor role of viscosity on the aerodynamic characteristics of flapping wings discussed previously.

With other contributors to $k$ being fixed (i.e. $k_{\text{flap}}$ and $k_{\text{ind}}$), and following Spedding & McArthur [2], an inviscid wing span efficiency owing to tip loss can be defined as the inverse of the induced power factor owing to tip loss

$$c_{\text{tip}} = \frac{1}{k_{\text{tip}\text{Ellington}}}.$$  

Figure 6 shows the effect of varying the Reynolds number on the value of this wing span efficiency for a fruitfly wing, with the mass and aspect ratio taken from table 1. For Reynolds numbers above approximately 100, the span efficiency is effectively independent of Reynolds number, whereas below this value, the efficiency drops increasingly rapidly towards zero. This is in good agreement with the experimental observations of Lentink & Dickinson [29].

For comparison, figure 6 also shows mean flapping cycle lift coefficient as a function of Reynolds number from CFD results of Wu & Sun [32]. Although the quantities shown in figure 6 are different, the span efficiency will directly influence the lift coefficient values attained during a flapping cycle, hence allowing a meaningful comparison of the shape of variation with Reynolds number.

4.2. Optimum flapping

We now derive conclusions from the proposed model regarding optimal wing planform shape and kinematics of hovering with $J_1 = 0$ from an engineering design point of view. This will be achieved by finding conditions that would minimize $k$ towards a unity value. Here, we introduce a simple geometrical function for the chord distribution that can be used for optimization studies. A number of wing stations along the wing length were selected and a simple optimization process was performed to obtain chord values that would minimize $k_{\text{ind}}$. Several function primitives matching the numerically obtained chord distribution were tested, and it was found that the following function provides a good compromise between accuracy and simplicity

$$c(r) = \frac{2\nu}{\pi} \text{ArcSech}(\hat{r}).$$  

A chord distribution defined by expression (4.3) for a wing with an aspect ratio of 4 (representing a mid-range value for insect wings [22]) gives a $k_{\text{ind}}$ value of 1.02 and is shown in figure 7a. This chord distribution is also very similar to the optimal hovering rotor shape, where the chord changes hyperbolically along the blade [14, p. 136]. Clearly, this distribution is physically difficult to realize in the root region and is seldom found in nature; however, it provides a useful reference point to which other less optimal, but more practical chord distributions can be compared. This approach has parallels with the use of the elliptic planform as an ideal minimum induced power baseline for fixed wing design (noting that the elliptic planform itself is impractical for most applications). As previously discussed, most insect wing shapes can be accurately represented using a beta
function with the non-dimensional radius of first moment of wing area varying between 40% and 60% of the wing length according to the insect species [22]. The chord distributions for these two extreme cases are shown in figure 7(a) for wings with the same aspect ratio and surface area as the arcsech case above. The value of $k_{ind}$ for the range between these two cases is shown in figure 7b. It is concluded that a wing whose centre of area location is at 0.4$R$ distance from the root has the lowest $k_{ind}$ of 1.07. This shows that this distribution is an efficient realistic planform with respect to $k_{ind}$. Note that an elliptic wing planform whose chord is varying like a half-ellipse along the wing length (figure 1) represents the optimum monoplane non-flapping fixed wing configuration with unity $k_{ind}$ value (case where $J_1$ approaches infinity). However, for the hovering flapping case, this wing gives a $k_{ind}$ of 1.13 (case where $J_1$ is zero).

It can be confirmed from the two presented tip loss models that normal hovering in which the two wing strokes provide weight support is more efficient with respect to tip losses (lower $k_{tip}$ value) than asymmetric strokes along an inclined stroke plane in which the weight support role is biased towards one stroke. This can be inferred directly from considering the ‘number of blades’ or the ‘frequency of lift impulses per wing beat frequency’ used for each case. Additionally, $k_{tip}$ will increase if an inclined stroke plane is employed. Clearly, $k_{tip}$ attains a unity value if a horizontal stroke plane and flapping angle amplitude of 90° are used. Hence, $k_{tip}$ describes behaviour consistent with previous optimization studies [33].

5. Conclusion

This paper has provided an analytical treatment of a flapping wing to capture the different aerodynamic effects influencing normal hovering flight in terms of the induced power factor. A number of non-ideal but physical effects that should be accounted when designing and/or analysing a hovering flapping wing are discussed comprehensively, including the effects of non-uniform downwash velocity distribution, tip losses and effective flapping disc area. A novel method that combines actuator disc and lifting line blade theories is proposed to handle the effect of non-uniform downwash distribution taking into consideration the possible effect of advance ratio on the aerodynamic characteristics of the wing. The developed model has been validated against results from the literature and good agreement with experimental investigations on the effect of advance ratio on the aerodynamics of a revolving fruitfly wing has been obtained. A very good agreement has also been found between the results of the proposed model and experimental measurements of the circulation distribution on a revolving fruitfly wing at zero advance ratio. Different methodologies for the evaluation of the tip losses are presented and analysed. This allowed an investigation on the variation of normal hovering flapping wings induced power factor with Reynolds number.

Specific conclusions for the evaluated induced power factor for eight hovering insect cases at zero advance ratio are as follows:

1) Contributions to the induced power factor: the non-uniform downwash effect leads to $k$-values ranging between 1.14 and 1.24; tip losses have been evaluated using two approaches where the two calculation results are around 1.1; losses owing to effective flapping disc area lead to $k$ around 1.2.

2) Overall induced power factor: values of the total $k$ accounting for all three discussed effects range from 1.30 for the ladybird and 1.88 for the hoverfly and are most clustered between 1.5 and 1.6 for the remaining species. Losses owing to reduction in effective actuation area from flapping stroke angle values less than $\pi$ represent the major contributor to the overall $k$ for most insects.

Specific conclusions for achieving ideal values of the hovering-induced power factor (i.e. $k = 1$) are as follows:

1) The contribution to $k$ arising from the non-uniform downwash effect is found to depend on both the wing chord distribution and the advance ratio. Thus, for a given advance ratio, this effect can be eliminated through appropriate choice of the wing chord distribution. For zero advance ratio (representing the typical hovering case which is of most interest), it is found that a chord distribution that follows an arcsech function can achieve this objective; however, the significant broadening of the chord at the root presents an implementation disadvantage that makes it difficult to realize this planform in practice. This chord distribution has strong similarity with the optimal distribution required for a rotary wing in hover.

2) An investigation into the effect of wing shape on induced power factor of relevance to real insects has been

![Figure 7. (a) Comparison of arcsech and beta chord distributions for hovering wings. (b) The induced power factor due to non-uniform downwash, $k_{ind}$, as a function of the non-dimensional radius of first moment of wing area. In both plots, the wing aspect ratio is 4 (mid-range value for insect wings) and the wing area is constant. (Online version in colour.)](http://rsif.royalsocietypublishing.org/Downloaded from http://rsif.royalsocietypublishing.org/)
undertaken using wing shapes represented by a beta function for a hovering condition at zero advance ratio. A wing planform whose centre of area is at 40% of the wing length provides the minimum k owing to non-uniform downwash with a value of 1.07.

(3) The tip losses contribution to k can be minimized through having lower disc loading values. Additionally, it is confirmed from the different tip loss models that normal hovering in which the two wing strokes provide weight support is more efficient than asymmetric strokes along an inclined stroke plane in which mainly one wing stroke provides the weight support.

(4) The effective flapping disc contribution to k can be controlled through the kinematic parameters: (i) stroke plane angle, and (ii) flapping stroke angle. Obviously, to minimize k_{tip} to unity, a horizontal stroke plane (i.e. normal hovering) should be used and a flapping stroke angle of 180° should be used.

References


